REGULARITY AND SPLITTING OF DIRECTED MINIMAL CONES

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Abstract. We show that directed minimal cones in $\mathbb{R}^{n+1}$ which have at most one singularity are – besides the trivial cases $\emptyset$, $\mathbb{R}^{n+1}$ – half spaces. Using blow-up techniques, this result can be used to get $C^{1,\lambda}$-regularity for the measure-theoretic boundary of almost minimal Caccioppoli sets which are representable as subgraphs in $\mathbb{R}^n$, $n \le 8$. This provides a different method to obtain a result due to De Giorgi. We also prove a splitting theorem for general directed minimal cones. Such a cone is of the form $\mathbb{R}^k \times C_0$, where C_0 is an undirected minimal cone or a half-line.

1. Introduction

Let $C \subset \mathbb{R}^{n+1}$ be a minimal cone with its vertex at the origin, i. e.

$$x \in C,\, \tau > 0 \quad \Longrightarrow \quad \tau x \in C$$

and

$$\int\limits_{\Omega} |D\chi_C| \le \int\limits_{\Omega} |D\chi_F|$$

whenever $\Omega \subset \mathbb{R}^{n+1}$ is an open bounded set and $C \bigtriangleup F \Subset \Omega$. We denote the characteristic function of a set E by χ_E. A measurable set $M \subset \mathbb{R}^{n+1}$ is called directed with respect to $w \in \mathbb{R}^{n+1} \setminus \{0\}$, if the functions

$$\mathbb{R} \ni t \mapsto D_\rho(M, x+tw) \equiv \omega_{n+1}^{-1} \rho^{-(n+1)} \int\limits_{B_\rho(x+tw)} \chi_M \equiv \int\limits_{B_\rho(x+tw)} \chi_M \Big/ \int\limits_{B_\rho} 1$$

are monotone decreasing for each $\rho > 0$ and $x \in \mathbb{R}^{n+1}$. Such a set is called directed. Now let us assume additionally that the cone C is directed, nontrivial ($C \neq \emptyset$ and $C \neq \mathbb{R}^{n+1}$) and has at most one singularity. In this case we prove a regularity theorem which states that such a cone is a half space. This result was also obtained by DeGiorgi in [1].

Even for non-directed cones, this result is well-known [7], provided that the dimension is small, i. e. $n \le 6$. Let $E \subset \mathbb{R}^{n+1}$ be an almost minimal set and $x \in \partial M$. We may assume that $x = 0$. Define for $t > 0$

$$E_t := \{x \in \mathbb{R}^{n+1} : tx \in E\}.$$

For an almost minimal set it is well known that there exists a sequence $t_i \downarrow 0$, such that

$$E_{t_i} \to C \text{ in } L^1_{loc}(\mathbb{R}^{n+1})$$

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and C is a nontrivial minimal cone. The regularity of ∂E at 0 is connected with the regularity of ∂E at 0 by the equivalence

$$0 \in \partial^\star E \quad \Longleftrightarrow \quad 0 \in \partial^\star C$$

for the reduced boundaries. Especially if ∂C is a hyperplane through the origin we get $0 \in \partial^\star E$. It is well-known [7] that in the case of small dimensions ($n \le 6$) this blow-up technique yields equality of the measure-theoretic boundary ∂E and the reduced boundary $\partial^\star E$ for almost minimal sets $E \subset \mathbb{R}^{n+1}$, because the only nontrivial minimal cones in $\mathbb{R}^{n+1}$ are half spaces. In $\mathbb{R}^8$ there are counterexamples.

An application of the regularity theorem is as follows: The regularity theorem allows us to conclude in the same way $\partial E = \partial^\star E$ in $\mathbb{R}^8$, when E is a directed set. This occurs for example when E is representable as a subgraph. Let $w \in \mathbb{R}^8 \setminus \{0\}$ be the corresponding direction. Obviously the sets E_{t_i} are again directed with respect to the same vector w. Since the limit in $L^1_{loc}(\mathbb{R}^{n+1})$ of sets which are directed with respect to w is directed, too, the blow-up technique yields a minimal nontrivial directed cone C. Thus ([5]) the dimension of the singular set is limited by

$$H^s(\partial E \setminus \partial^\star E) = 0 \quad \forall s > n - 7.$$

If there were another singular point apart from the origin, the property of C being a cone would imply $H^1(\partial E \setminus \partial^\star E) = +\infty$, contradicting $n \le 7$. This shows that there is no singular point besides the origin. Therefore the regularity theorem implies that C is a half space and consequently we get $\partial E = \partial^\star E$. This proof is valid for all $n \le 7$. The problem in extending this result to $\mathbb{R}^{n+1}$ with $n \ge 8$ is to show that the singular set consists of at most one point. In $\mathbb{R}^9$ there is indeed a counterexample which shows that there exists a nontrivial minimal directed cone which is different from a half space.

We give an alternative approach to these results that yields more information about the structure of a directed minimal cone. This is contained in the splitting theorem that generalizes Corollary 4.10, namely, if C is an open singular minimal cone with its vertex at the origin, then C is of the form $C_0 \times \mathbb{R}^k$, where C_0 is an undirected minimal cone.

This rest of the paper is organized as follows: In Section 2 we introduce notations and recall some well known facts. In Section 3 we consider directed cones and show that directed cones are weakly star-shaped with respect to certain points. These properties are essential for the proof of the regularity theorem which is stated and proved in Section 4. In Section 7 we state and prove the splitting theorem. In Section 5 we apply the regularity theorem to subgraphs, in Section 6 we show how it can be applied to getting hypersurfaces of prescribed mean curvature homeomorphic to S^n for $n \le 7$. Since we use the fact that the cones arising are directed, the results of [2] for up to seven dimensions remain true in eight dimensions.

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2. Preliminaries

Notation 2.1. The characteristic function of a set M will be denoted by χ_M. If we are concerned with coordinates in $\mathbb{R}^{n+1}$, (x, t) stands for $(x^1, \ldots, x^n, t)$ and $x = (\hat{x}, x^{n+1})$ abbreviates the first n coordinates by $\hat{x}$. Assume in the whole paper $n \ge 2$.

Definition 2.2 (almost minimal)**.** *A measurable set $E \subset \mathbb{R}^{n+1}$ is called almost minimal in $\Omega \subset \mathbb{R}^{n+1}$, if there exists a λ such that $0 < \lambda < 1$ and for all sets $A \Subset \Omega$ there are numbers R and K such that $0 < R < \operatorname{dist}(A, \complement\Omega)$ and $K \geq 0$ which fulfill the inequality*

$$\int_{B_\rho(x)} |D\chi_E| \leq \int_{B_\rho(x)} |D\chi_F| + K\rho^{n+2\lambda}$$

for every $x \in A$, $0 < \rho < R$ and $E \triangle F \Subset B_\rho(x)$.

E is called (locally) minimal, if we can choose $K = 0$.

In this paper we deal with two kinds of boundaries, the measure-theoretical boundary and the reduced boundary, which are defined according to [5, p. 43]. Obviously these definitions are invariant, if we change the respective sets by a set of Lebesgue measure zero.

Definition 2.3 ((measure-theoretical) boundary)**.** *Let $E \subset \mathbb{R}^{n+1}$ be a measurable set. Define the measure-theoretical boundary*

$$\partial E := \{x \in \mathbb{R}^{n+1} : 0 < |B_r(x) \cap E| < |B_r(x)| \quad \forall r > 0\},$$

and the measure-theoretical interior and the measure-theoretical exterior by

$$E_\mu := \{x \in \mathbb{R}^{n+1} : \exists r > 0 : |B_r(x) \cap E| = |B_r(x)|\}$$

and

$$\complement_\mu E := \{x \in \mathbb{R}^{n+1} : \exists r > 0 : 0 = |B_r(x) \cap E|\},$$

respectively.

According to this definition, E_μ and $\complement_\mu E$ are open sets. $\mathbb{R}^{n+1}$ is the disjoint union of E_μ, $\complement_\mu E$ and ∂E. Later-on we will denote E_μ resp. $\complement_\mu E$ simply by E resp. $\complement E$. If E is an almost minimal set, E and E_μ lie in the same equivalence class. This follows immediately, since the boundary of E is the union of a differentiable manifold and of a set of H^s measure zero, if $s > n - 7$, and is therefore a set of H^{n+1} measure zero. So we will always assume that an almost minimal set is open.

Definition 2.4 (reduced boundary)**.** *Let $E \subset \mathbb{R}^{n+1}$ be a Caccioppoli set. The reduced boundary $\partial^\star E$ consists of those points $x \in \mathbb{R}^{n+1}$, for which*

$$\int_{B_\rho(x)} |D\chi_E| > 0 \quad \forall \rho > 0$$

is valid and the limit

$$\lim_{\rho \downarrow 0} \nu_\rho(x) = \nu(x)$$

exists. The vector $\nu_\rho(x)$ is defined by

$$\nu_\rho(x) := \frac{\int_{B_\rho(x)} D\chi_E}{\int_{B_\rho(x)} |D\chi_E|}.$$

If $|\nu(x)| = 1$, $\nu(x)$ is called the inner (unit) normal to E at x.

Definition 2.5 (singularity). *A point $x \in \mathbb{R}^{n+1}$ is called a singularity of the measurable set $M \subset \mathbb{R}^{n+1}$, if $x \in \partial M \setminus \partial^\star M$. The set $\partial M \setminus \partial^\star M$ is called the singular set of M or the singular set of ∂M.*

A point in the boundary which is not singular is called regular.

Definition 2.6 (cone). *A set $C \subset \mathbb{R}^{n+1}$ is called a cone with vertex x, if*

$$y \in C \quad \Longrightarrow \quad x + \tau(y-x) \in C \quad \forall 0 < \tau.$$

Moreover, if $C \neq \emptyset$ and $C \neq \mathbb{R}^{n+1}$, C is called a nontrivial cone.

Lemma and Definition 2.7. *Let $E \subset \mathbb{R}^{n+1}$ be an almost minimal Caccioppoli set in the open set $\Omega \subset \mathbb{R}^{n+1}$. Assume that $0 \in \partial E$. If $E_t := \{x \in \mathbb{R}^{n+1} : tx \in E\}$ converges in $L^1_{loc}(\mathbb{R}^{n+1})$ to a set C for a given sequence $t_i \to 0$, $t_i > 0$, then C is a cone which is different from $\mathbb{R}^{n+1}$ and $\emptyset$. We define L^1-convergence of sets by using the corresponding characteristic functions. Such a cone C is called a blow-up cone of E around 0.*

Proof: Proceed as in [5, Theorem 9.3] and use the estimates of [7, p. 118] and [7, Proposition, p. 137]. □

Remark 2.8. In small dimensions, i. e. for $n+1 \le 8$, the blow-up cone C of an almost minimal set around a point of its boundary has no singularity apart from the origin: If there were another singularity, this would imply that a half-line would be singular, because C is a cone. But this is a contradiction to the fact, that $H^s(\partial^\star C \setminus \partial C) = 0$ for $s > n-7$.

Recall that the reduced boundary of a minimal set is an analytic manifold. So, in particular, the boundary of a minimal cone is analytic apart from the origin, if $n+1 \le 8$.

Definition 2.9 (subgraph). *Let $\varphi : A \to [-\infty, +\infty]$ be a measurable function. Define the subgraph of the function φ by*

$$\operatorname{sub}\varphi := \{(x,t) \in A \times \mathbb{R} : t < \varphi(x)\}.$$

Remark 2.10. If a cone with its vertex at the origin is a subgraph of a function u, then u is positive homogeneous of degree 1.

3. Directed Cones

Definition 3.1 (directed set). *Let $E \subset \mathbb{R}^{n+1}$ be a measurable set, $y \in \mathbb{R}^{n+1} \setminus \{0\}$.*

$$D_\rho(E,x) := \omega_{n+1}^{-1} \rho^{-n-1} \int\limits_{B_\rho(x)} \chi_E$$

is called approximative density of E in x with respect to the radius ρ.

A set E is called directed with respect to y, if the maps

$$f_{x,\rho} : \mathbb{R} \to [0,1], \quad t \mapsto D_\rho(E, x+ty)$$

are monotone decreasing with respect to t for any $x \in \mathbb{R}^{n+1}$, $\rho > 0$. Such y is called direction of the set E.

It is possible that a set has several linearly independent directions.

For a measurable set M we have the definition (cf. e. g. [5])

$$x \in M :\Longleftrightarrow \exists \rho > 0 : \int\limits_{B_\rho(x)} \chi_M = \int\limits_{B_\rho(x)} 1 \Longleftrightarrow \exists \rho > 0 : D_\rho(M, x) = 1,$$

$$x \in \complement M :\Longleftrightarrow \exists \rho > 0 : \int\limits_{B_\rho(x)} \chi_M = 0 \Longleftrightarrow \exists \rho > 0 : D_\rho(M, x) = 0.$$

Lemma 3.2. *Let $M \subset \mathbb{R}^{n+1}$ be a measurable set which is directed with respect to v. Then for any $x \in \mathbb{R}^{n+1}$ and $t > 0$ we have the implications*

$$x \in \complement M \quad \Longrightarrow \quad x + tv \in \complement M$$

and

$$x \in M \quad \Longrightarrow \quad x - tv \in M.$$

Proof: According to the definition of the measure-theoretical complement of a set there exists in the case $x \in \complement M$ a $\rho > 0$ such that $D_\rho(M, x) = 0$. Since M is directed with respect to v, it follows for $t > 0$ that

$$D_\rho(M, x + tv) \leq D_\rho(M, x) = 0.$$

$D_\rho(\cdot, \cdot)$ is non-negative, so $D_\rho(M, x + tv) = 0$. This implies $x + tv \in \complement M$.

In the case $x \in M$ the proof is similar: $x \in M \Rightarrow \exists \rho > 0 : D_\rho(M, x) = 1$. M is a directed set. Now $t > 0$ implies $D_\rho(M, x - tv) \geq D_\rho(M, x) = 1$ and it follows that $x - tv \in M$ as above. □

Corollary 3.3. *Let $M \subset \mathbb{R}^{n+1}$ be a measurable set which is directed with respect to v. If $x \in \mathbb{R}^{n+1}$ and $t > 0$ are such that $x \in \partial M$ and $x + tv \in \partial M$, it follows that $x + \tau v \in \partial M$ for $0 \leq \tau \leq t$.*

Lemma 3.4. *Let $M \subset \mathbb{R}^{n+1}$ be a measurable set, directed with respect to v. For $x_0 \in \partial M$ exactly one of the following possibilities occurs:*

(1) $\exists t \neq 0 : x_0 + tv \in \partial M$,
(2) $\forall t > 0 : x_0 + tv \in \complement M$ *and* $x_0 - tv \in M$.

Proof: Assume $x_0 + tv \notin \partial M$ for $t \neq 0$, i. e. $x_0 + tv \in M \cup \complement M$ for $t \neq 0$. Therefore we have to show that the second possibility occurs:

For $t > 0$ the possibility $x_0 + tv \in M$ is excluded: Assume $t > 0$. Then $x_0 + tv \in M$ cannot happen, because in accordance to Lemma 3.2 $x_0 + tv \in M$ implies $x_0 = (x_0 + tv) - tv \in M$ contradicting $x_0 \in \partial M$. It follows $x_0 + tv \in \complement M$ for $t > 0$. $x_0 - tv \in \complement M$ does not occur for $t > 0$ for a similar reason. Thus the statement is proved. □

Lemma 3.5. *Let $M \subset \mathbb{R}^{n+1}$ be a measurable set which is directed with respect to v. Then $\complement M$ and $-M \equiv \{x \in \mathbb{R}^{n+1} : -x \in M\}$ are directed with respect to $-v$. Especially $-\complement M$ is again directed with respect to v.*

Proof: The statement follows at once from the equations

$$D_\rho(M, y) + D_\rho(\complement M, y) = 1$$

and

$$D_\rho(-M, -x - \tau v) = D_\rho(M, x + \tau v)$$

which are valid for any $x \in \mathbb{R}^{n+1}$ and $\rho > 0$. □

Lemma 3.6. *Let $E \subset \mathbb{R}^{n+1}$ be representable as a subgraph. If $t_i \downarrow 0$ is as before such that $E_{t_i} \equiv \{x \in \mathbb{R}^{n+1} : t_i x \in E\}$ converges in $L^1_{loc}(\mathbb{R}^{n+1})$ to a cone C, then C is directed with respect to e_{n+1}.*

Proof: As a subgraph, E is directed with respect to e_{n+1} (cf. Remark 3.9). If C were not directed with respect to e_{n+1}, there would be $x \in \mathbb{R}^{n+1}$, $\rho > 0$ and $t > 0$ such that

$$\int\limits_{B_\rho(x)} \chi_C < \int\limits_{B_\rho(x+te_{n+1})} \chi_C.$$

In the same way as for E we get that E_{t_i} is directed with respect to e_{n+1}. Because of the convergence $E_{t_i} \to C$ in $L^1_{loc}(\mathbb{R}^{n+1})$ we immediately get a contradiction to the inequality above. $\square$

Definition 3.7 (weakly star-shaped)**.** *A set $M \subset \mathbb{R}^{n+1}$ is called weakly star-shaped with center x, if for all $z \in M$ we have $x + \tau(z - x) \in M$ if $0 < \tau \le 1$.*

The following Lemma is essential for the proof of the regularity theorem:

Lemma 3.8. *Let C be an open cone with vertex at the origin which is directed with respect to e_{n+1}. Then C is weakly star-shaped with center $x = (0, -t)$ for all $t > 0$.*

Proof: Let $y = (\hat{y}, y^{n+1}) \in C$ and τ with $0 < \tau < 1$ be arbitrary. We show that $x + \tau(y - x) = (\tau\hat{y}, -t + \tau(y^{n+1} + t))$ is an element of C. Being a cone, C contains $(\tau\hat{y}, \tau y^{n+1})$ because of $y \in C$. $-t(1-\tau)$ is negative, so Lemma 3.2 implies $(\tau\hat{y}, \tau y^{n+1}) - t(1-\tau)(0, 1) = (\tau\hat{y}, -t + \tau(y^{n+1} + t)) \in C$. $\square$

Remark 3.9. For a measurable set $M \subset \mathbb{R}^{n+1}$ such that $H^{n+1}(\partial M) = 0$ the following two statements are equivalent:

(1) M is the subgraph of a measurable function u,
(2) M is directed with respect to e_{n+1}.

The measurable function u in (i) is given by

$$u(\hat{x}) := \sup\{t \in \mathbb{R} : (\hat{x}, t) \in M\}.$$

Proof: "(i)$\Longrightarrow$(ii)":
Let $M = \operatorname{sub} u$ be a given set. It follows that $\chi_M(x, t) = 1$ for $t < u(x)$ and $\chi_M(x, t) = 0$ for $t > u(x)$, i. e. $\chi_M(x, t)$ is monotone decreasing with respect to t for any $x \in \mathbb{R}^n$. By integrating, we get for any $\rho > 0$, $\tau > 0$ and $x \in \mathbb{R}^{n+1}$

$$\int\limits_{B_\rho(x)} \chi_M(z)dz \ge \int\limits_{B_\rho(x)} \chi_M(z + \tau e_{n+1})dz = \int\limits_{B_\rho(x+\tau e_{n+1})} \chi_M(y)dy.$$

Therefore M is a directed set.
"(ii)$\Longrightarrow$(i)":
Define for $\hat{x} \in \mathbb{R}^n$

$$u(\hat{x}) := \sup\{t \in \mathbb{R} : (\hat{x}, t) \in M\}.$$

We remark that $(\hat{x}, t) \in M$ is equivalent to the existence of a $\rho > 0$ such that $D_\rho(M, (\hat{x}, t)) = 1$. Since M is a measurable set the function u is measurable. Define $U := \operatorname{sub} u$.

a) $M \subset U$:
Assume that $(\hat{x}, t) \in M$. Since the supremum in the definition of $u(\hat{x})$ is not assumed, it follows $u(\hat{x}) > t$ and therefore $(\hat{x}, t) \in U$.

b) $U \subset M$:
Assume $(\hat{x}, t) \in U$, i. e. $u(\hat{x}) > t$. The definition of $u(\hat{x})$ implies that there exists a τ such that $u(\hat{x}) > \tau > t$ and $(\hat{x}, \tau) \in M$. Now Lemma 3.2 yields $(\hat{x}, t) \in M$, since $t - \tau < 0$ and $(\hat{x}, \tau) + (t - \tau)(0, 1) = (\hat{x}, t)$. □

Definition 3.10 (cone of directions)**.** *Let $E \subset \mathbb{R}^{n+1}$ be a measurable set.* Dir (E), *the cone of directions, is defined to be the set of all directions of E together with the origin.*

Remark 3.11. Definition 3.10 is equivalent to

$$\begin{aligned} \operatorname{Dir}(E) :=& \{y \in \mathbb{R}^{n+1} : f_{x,\rho} : \mathbb{R} \to [0,1], t \mapsto D_\rho(E, x + ty) \\ & \text{is monotone decreasing for any } x \in \mathbb{R}^{n+1} \text{ and any } \rho > 0\}. \end{aligned}$$

Lemma 3.12. *Let $E \subset \mathbb{R}^{n+1}$ be a measurable set. Then* Dir (E) *is a closed cone which is also closed under addition.*

Proof:
(i) The fact, that Dir (E) is a cone follows immediately from the Definition 3.10.
(ii) Dir (E) is closed under addition:
Let $y_1, y_2 \in \operatorname{Dir}(E)$ be arbitrary. According to the definition this is equivalent to

$$D_{\rho_1}(E, x_1 + t_1 y_1) \leq D_{\rho_1}(E, x_1 + \tau_1 y_1)$$

and

$$D_{\rho_2}(E, x_2 + t_2 y_2) \leq D_{\rho_2}(E, x_2 + \tau_2 y_2)$$

for any $x_1, x_2 \in \mathbb{R}^{n+1}$, $\rho_1, \rho_2 > 0$ and $t_1, t_2, \tau_1, \tau_2 \in \mathbb{R}$ such that $t_1 \geq \tau_1$ and $t_2 \geq \tau_2$. We have to show now, that

$$D_\rho(E, x + t(y_1 + y_2)) \leq D_\rho(E, x + \tau(y_1 + y_2))$$

for any $x \in \mathbb{R}^{n+1}$, $\rho > 0$ and $t, \tau \in \mathbb{R}$ such that $t \geq \tau$. We choose now $x_1 = x + ty_2$, $t_1 = t$, $\rho_1 = \rho$, $\tau_1 = \tau$, $x_2 = x + \tau y_1$, $t_2 = t$, $\rho_2 = \rho$, $\tau_2 = \tau$ and deduce from the inequalities above

$$\begin{aligned} D_\rho(E, (x + ty_2) + ty_1) \leq& D_\rho(E, (x + ty_2) + \tau y_1) \\ =& D_\rho(E, (x + \tau y_1) + ty_2) \\ \leq& D_\rho(E, (x + \tau y_1) + \tau y_2) \end{aligned}$$

verifying the claimed inequality.
(iii) Dir (E) is a closed set:
Let $y_i \in \operatorname{Dir}(E)$ for $i \in \mathbb{N}$ such that $y_i \to y$ as $i \to \infty$. We have to show $y \in \operatorname{Dir}(E)$. Assume in contrast $y \notin \operatorname{Dir}(E)$. Then there exists $x \in \mathbb{R}^{n+1}$, $\rho > 0$, $t, \tau \in \mathbb{R}$ such that $t \geq \tau$ and

$$D_\rho(E, x + ty) - D_\rho(E, x + \tau y) > 0.$$

W. l. o. g. we can assume $\tau = 0$. $y_i \in \operatorname{Dir}(E)$ implies

$$D_\rho(E, x + ty_i) - D_\rho(E, x) \leq 0.$$

Since $D_\rho(E, x + ty_i)$ converges to $D_\rho(E, x + ty)$ as i tends to infinity we get a contradiction and the statement follows. □

4. Regularity Theorem

Definition 4.1. *An open set $M \subset \mathbb{R}^{n+1}$ is said to lie on one side of a hyperplane T if an adjusted rotation and translation of the coordinate system yields the following situation*

$$\begin{aligned} x = (\hat{x}, x^{n+1}) \in T \quad &\Longleftrightarrow \quad x^{n+1} = 0, \\ x = (\hat{x}, x^{n+1}) \in M \quad &\Longrightarrow \quad x^{n+1} > 0. \end{aligned}$$

Lemma 4.2. *Let $C \subset \mathbb{R}^{n+1}$ be a cone which is representable as a subgraph of a C^1-function u in a neighborhood of its vertex. Then ∂C and the tangential hyperplane T to ∂C at the vertex of C coincide and C is a half space.*

Proof: By a translation we can assume w. l. o. g. that the vertex of the cone is the origin. Since C is a cone, u is positive homogeneous of degree 1 and has a well-defined extension (w. l. o. g. u) of the same homogenity which is defined on the whole $\mathbb{R}^n$. The subgraph of u is C. Let $v \in \mathbb{R}^n$ be arbitrary. It follows

$$\langle Du(0), v\rangle = \lim_{t\to 0} \frac{u(tv) - u(0)}{t}.$$

If we take into account $u(0) = 0$ and use the fact that u is a positive homogeneous function, we deduce for $t > 0$

$$\langle Du(0), v\rangle = \lim_{t\to 0} \frac{t \cdot u(v)}{t} = u(v).$$

The left-hand side of this equality is linear with respect to v. Hence u is a linear function. Thus C is a subgraph of a linear function and the statement follows. $\square$

Lemma 4.3. *Let $C \subset \mathbb{R}^{n+1}$ be an open minimal cone with its vertex at the origin. If C is directed with respect to e_{n+1} and there exists a $t > 0$ such that $z := (0, t) \in \complement C$ and $(0, -t) \in C$, then ∂C is a hyperplane.*

Proof: W. l. o. g. we assume $n \geq 7$. Choose $r > 0$ such that $D_r(C, z) = 0$ and $D_r(C, -z) = 1$. Define a cone K by

$$K := \{(\hat{x}, -\tau) : \tau > 0, \tau r > |\hat{x}| t\}.$$

It follows

(i) $K \subset C$ and $-K \subset \complement C$:

Let $(\hat{x}, -\tau) \in K$ be arbitrary. Since K is a cone, we get $\left(\hat{x}\frac{t}{\tau}, -\tau\frac{t}{\tau}\right) = \left(\hat{x}\frac{t}{\tau}, -t\right) \in K$. Now $\left|\hat{x}\frac{t}{\tau}\right| < r$ implies $\left(\hat{x}\frac{t}{\tau}, -t\right) \in B_r(-z)$. Choose $s > 0$ such that $B_s\left(\left(\hat{x}\frac{t}{\tau}, -t\right)\right) \subset B_r(-z)$. The equality $D_r(C, -z) = 1$ implies $D_s\left(C, \left(\hat{x}\frac{t}{\tau}, -t\right)\right) = 1$ and therefore we get $\left(\hat{x}\frac{t}{\tau}, -t\right) \in C$. C is a cone, so we deduce $(\hat{x}, -\tau) \in C$. As $(\hat{x}, -\tau) \in K$ was arbitrary, so we get $K \subset C$.

In the same way $-K := \{(\hat{x}, \tau) : \tau > 0, \tau r > |\hat{x}| t\} \subset \complement C$ is proved.

(ii) Representability of ∂C as a graph:

We will show, that ∂C can be represented as a subgraph over $\mathbb{R}^n \setminus \Sigma$ besides a set which has H^{n-5}-measure zero. Σ is a closed set of H^{n-6}-measure zero.

For $s > n - 7$ we get $H^s(\partial C \setminus \partial^\star C) = 0$ by Theorem [5, Theorem 11.8, p. 134]. This implies especially $H^{n-6}(\partial C \setminus \partial^\star C) = 0$. Define $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ by $\pi((\hat{x}, x^{n+1})) = \hat{x}$ and $\Sigma := \pi(\partial C \setminus \partial^\star C)$. Since π is Lipschitz continuous we deduce $H^{n-6}(\Sigma) = 0$, hence $H^{n-5}(\Sigma \times \mathbb{R}) = 0$ and $H^{n-5}((\Sigma \times \mathbb{R}) \cap \partial C) = 0$. Σ is a closed set, because π is continuous and in view of (i) $(\partial C \setminus \partial^\star C) \cap (F \times \mathbb{R})$ is compact for any compact set $F \subset \mathbb{R}^n$.

Let $\hat{x} \in \mathbb{R}^n \setminus \Sigma$ be arbitrary. As $K \subset C$, $\tau > \frac{|\hat{x}|t}{r}$ implies $(\hat{x}, \tau) \in \complement C$ and $(\hat{x}, -\tau) \in C$. $\{\hat{x}\} \times \mathbb{R}$ is connected, C, $\complement C$ und ∂C are disjoint and we have $C \cup \complement C \cup \partial C = \mathbb{R}^n$. Regarding the fact, that C and $\complement C$ are open sets we deduce $(\{\hat{x}\} \times \mathbb{R}) \cap \partial C \neq \emptyset$, because a connected set is not the disjoint union of two non-empty open sets, in this case $(\{\hat{x}\} \times \mathbb{R}) \cap C$ with $(\hat{x}, -\tau) \in C$ and $(\{\hat{x}\} \times \mathbb{R}) \cap \complement C$ with $(\hat{x}, \tau) \in \complement C$ for all $\tau > \frac{|\hat{x}|t}{r}$. The set $(\{\hat{x}\} \times \mathbb{R}) \cap \partial C$ is bounded because $(\hat{x}, \tau) \in \complement C$ and $(\hat{x}, -\tau) \in C$ for all $\tau > \frac{|\hat{x}|t}{r}$. C is directed with respect to e_{n+1} and ∂C is closed, so Lemma 3.2 implies $(\{\hat{x}\} \times \mathbb{R}) \cap \partial C = \{\hat{x}\} \times I$ for a compact interval I. The boundary of C is analytic in the complement of $\Sigma \times \mathbb{R}$ and so I consists of exactly one point. Therefore we have a function $u \in L^1_{loc}(\mathbb{R}^n \setminus \Sigma)$, whose graph coincides with $\partial C \setminus (\Sigma \times \mathbb{R})$ and we have the equality $C \cap ((\mathbb{R}^n \setminus \Sigma) \times \mathbb{R}) = \operatorname{sub} u$. Observe, however, that u is not automatically analytic, because the boundary ∂C in the complement of $\Sigma \times \mathbb{R}$ is only analytic as a manifold.

(iii) Regularity of u:
Let $\Omega \Subset \mathbb{R}^n \setminus \Sigma$ be an open ball. Since the subgraph of $u|_\Omega$ has finite perimeter in $\Omega \times \mathbb{R}$, i. e. $\int\limits_{\Omega \times \mathbb{R}} |D\chi_{\operatorname{sub} u}| < \infty$, and $u \in L^\infty(\Omega)$, we can use [4, Theorem 1, p. 317] to deduce $u \in BV(\Omega)$. According to [3] we get $u \in C^{0,1}(\Omega)$. But u is also a weak solution of the minimal surface equation and this implies $u \in C^2(\Omega)$. $\Omega \Subset \mathbb{R}^n \setminus \Sigma$ was an arbitrary open ball. Thus we deduce $u \in C^2(\mathbb{R}^n \setminus \Sigma)$. We have $H^{n-6}(\Sigma) = 0$, so Theorem [5, Theorem 16.9] can be applied, i. e. u can be extended to a function in $C^2(\mathbb{R}^n)$ solving the minimal surface equation. $\operatorname{sub} u$ is an element of the L^1-equivalence class of C independent of the choice of the extension, because another extension changes $\operatorname{sub} u$ at most by a subset of $\Sigma \times \mathbb{R}$ and $H^{n-5}(\Sigma \times \mathbb{R}) = 0$.

(iv) ∂C is a hyperplane:
This follows immediately from Lemma 4.2. □

The following Lemma is - besides Lemma 3.8 - the essential part of the proof of the regularity theorem:

Lemma 4.4. *Let $M \subset \mathbb{R}^{n+1}$ be an open weakly star-shaped set with center $x_0 \in \partial M$. If the boundary of M is of class C^1 in a neighborhood of x_0, then M lies on one side of the tangential hyperplane $T = T_{x_0}\partial M \subset \mathbb{R}^{n+1}$.*

Proof: By a translation of the coordinates we can assume that $x_0 = (0, 0) \in \mathbb{R}^n \times \mathbb{R}$. Since ∂M is of class C^1 in a neighborhood of x_0, we reach the following situation by a suitable rotation of the coordinates: There exist $R > 0$ and $u \in C^1(\hat{B}_{2R}) \equiv C^1(\{\hat{x} \in \mathbb{R}^n : |\hat{x}| < 2R\})$ such that $u(0) = 0$, $Du(0) = 0$ and for any $x = (\hat{x}, x^{n+1}) \in \hat{B}_R \times (-R, R)$ we have the three equivalences

$$x \in \partial M \iff x^{n+1} = u(\hat{x}),$$
$$x \in M \iff x^{n+1} > u(\hat{x})$$

and

$$x \in \complement M \iff x^{n+1} < u(\hat{x}).$$

By construction we have for any $x \in \mathbb{R}^{n+1}$

$$x \in T \iff x^{n+1} = 0.$$

Assume now, the assertion would be false, more precisely: The chosen coordinates were not fulfilling the conditions of Definition 4.1. That is, the condition

$$x = (\hat{x}, x^{n+1}) \in M \implies x^{n+1} > 0$$

is violated for an $x \in \mathbb{R}^{n+1}$. As M is an open set, there exists a point $y_0 = (\hat{y}_0, y_0^{n+1}) \in M$ with $y_0^{n+1} < 0$. First, we observe that $\hat{y}_0 = 0$ cannot occur, for M is weakly star-shaped with center $x_0 = (0,0)$ and we have for $y_0 \in M$ that τy_0 is in M for $0 < \tau \le 1$. This implies for $|\tau y_0| < R$, i. e. for sufficiently small $\tau > 0$, we get the inequality

$$\tau y_0^{n+1} < 0 = u(0) = u(\hat{y}_0)$$

which contradicts the assumption

$$x \in M \quad \Longleftrightarrow \quad x^{n+1} > u(\hat{x})$$

for $x \in \hat{B}_R \times (-R, R)$. Assume therefore $\hat{y}_0 \neq 0$. Using again the fact that M is weakly star-shaped, we get $\tau y_0 = (\tau\hat{y}_0, \tau y_0^{n+1}) \in M$ for $0 < \tau \le 1$. Choose now $\varepsilon > 0$ such that

$$0 < \tau < \varepsilon \quad \Longrightarrow \quad \tau y_0 \in \hat{B}_R \times (-R, R).$$

In $\hat{B}_R \times (-R, R)$, the set M was characterized by the equivalence

$$x = (\hat{x}, x^{n+1}) \in M \quad \Longleftrightarrow \quad x^{n+1} > u(\hat{x}).$$

This implies $u(\tau\hat{y}_0) < \tau y_0^{n+1}$ for $0 < \tau < \varepsilon$. If we take into account that $u(0) = 0$, $|\hat{y}_0| \neq 0$ and $y_0^{n+1} < 0$, we deduce

$$u(\tau\hat{y}_0) < \tau y_0^{n+1} < 0,$$
$$|u(\tau\hat{y}_0) - u(0)| = -u(\tau\hat{y}_0) > -\tau y_0^{n+1},$$

and finally

$$\left|\frac{u(\tau\hat{y}_0) - u(0)}{\tau|\hat{y}_0|}\right| > \frac{-y_0^{n+1}}{|\hat{y}_0|}$$

for all $0 < \tau < \varepsilon$. The right-hand side of this inequality is independent of τ and positive, so we get a contradiction for $\tau \to 0$ to the fact that the coordinates were chosen such that $Du(0) = 0$. □

Theorem 4.5 (Regularity Theorem). *Let $C \subset \mathbb{R}^{n+1}$ be an open nontrivial minimal cone which is directed with respect to e_{n+1}. Assume that the vertex of C is the origin and that C has at most one singularity, i. e. C is at most singular in the origin. Then ∂C is a hyperplane and C is a half space.*

For $n \le 6$ there is nothing to be proved, the statement is true according to [5, Theorem 10.11, p. 127] even for non-directed cones.

Proof: ∂C is regular apart from the origin, so $\partial C \setminus \{0\}$ is analytic.

In view of Lemma 4.3 and due to the fact that C is directed, it suffices to consider the case that $x_0 = (0, t) \in \partial C$ for some $t \neq 0$.

We may assume that $t < 0$. Otherwise consider $-\complement C \equiv \{x \in \mathbb{R}^{n+1} : -x \in \complement C\}$ instead of C. According to Lemma 3.5, this set is directed with respect to e_{n+1}.

Now Lemma 3.8 implies that C is weakly star-shaped with center x_0. ∂C is regular apart from the origin. Therefore we have a well-defined tangential hyperplane T at ∂C. With the help of Lemma 4.4 we conclude, that C lies on one side of T. Since C is nontrivial, we get $0 \in \partial C$. Finally [5, Theorem 15.5, p. 174] implies that C is a half space and $\partial C = T$ is a hyperplane. □

Corollary 4.6. *Let $C \subset \mathbb{R}^{n+1}$ be an open nontrivial minimal cone which is directed with respect to e_{n+1}. If $n + 1 \le 8$ then ∂C is a hyperplane and C is a half space.*

Proof: Remark 2.8 guarantees that C has at most one singularity which is at the vertex if it exists. Thus our Regularity Theorem 4.5 yields the statements. □

Remark 4.7. In the Regularity Theorem 4.5, we assumed that C is regular outside its vertex. In the proof, however, we use only the fact, that there is a $t > 0$ which satisfies the following two conditions:

(i) $(0,t) \in \partial C \implies \partial C$ is a C^1-manifold in a neighborhood of $(0,t)$.

(ii) $(0,-t) \in \partial C \implies \partial C$ is a C^1-manifold in a neighborhood of $(0,-t)$.

Therefore we get the following corollary:

Corollary 4.8. *Let $C \subset \mathbb{R}^{n+1}$ be an open nontrivial minimal cone which is directed with respect to e_{n+1} and which satisfies the two conditions (i) and (ii) stated above for a positive $t > 0$. (If one of the points $(0,t)$ or $(0,-t)$ is an element of ∂C, we need in fact only the respective condition.) Then C is a half space.*

Remark 4.9. Conditions (i) and (ii) in Corollary 4.8 can alternatively be replaced by one of the following conditions:

(i) There is a direction w of C which fulfills $w \notin \Sigma$ and $-w \notin \Sigma$ or $w \in \partial C \setminus \Sigma$ or $-w \in \partial C \setminus \Sigma$.

(ii) There is one direction w of C which fulfills $w \notin \langle \Sigma \rangle$.

(iii) $\dim\langle D \rangle > \dim\langle \Sigma \rangle$.

Here D is the set of all directions and Σ is the singular set of C. The brackets denote the vectorspace which is spanned by the respective set.

Proof: This follows at once from Corollary 4.8, if we use the fact, that $\partial C \setminus \Sigma$ is an open C^1-manifold. □

Corollary 4.10. *Let $C \subset \mathbb{R}^{n+1}$ be an open nontrivial minimal cone with vertex at the origin. Suppose C has k lineareły independent directions, $k \in \mathbb{N}$. Then C is a half-space provided that $n + 1 - k \le 7$ or $H^k(\partial C \setminus \partial^\star C) = 0$.*

Proof: In the first case ($n+1-k \le 7$) we get $H^k(\partial C \setminus \partial^\star C) = 0$, for $k > n-7$ and C is an almost minimal set, so the second case includes the first one.

Since C has k linearely independent directions we deduce according to Lemma 3.12 that $H^k(\mathrm{Dir}\,(C)) = \infty$. But we assumed that the k-dimensional Hausdorff-measure of the singular set of C vanishes. So we can find $x \in \mathbb{R}^{n+1} \setminus \{0\}$ such that $x \notin \partial C \setminus \partial^\star C$, $-x \notin \partial C \setminus \partial^\star C$ and $x \in \mathrm{Dir}\,(C)$. Then we apply Corollary 4.8. □

5. Applications

Theorem 5.1 (regularity for subgraphs)**.** *Assume that $E \subset \mathbb{R}^{n+1}$, $n \le 7$, is a subgraph and almost minimal (with constant λ) in Ω, $\Omega \subset \mathbb{R}^{n+1}$ open, then we get $\partial E \cap \Omega = \partial^\star E \cap \Omega$ and $\partial E \cap \Omega$ is a $C^{1,\lambda}$-manifold.*

Proof: According to the definitions we get $\partial^\star E \subset \partial E$. Let $x_0 \in \partial E \cap \Omega$ be an arbitrary point. Show $x_0 \in \partial^\star E$: By virtue of a translation we can assume that $x_0 = 0$. Then [7, Proposition, p. 137] guarantees that there is a sequence $t_i \to 0$, $t_i > 0$, such that $E_{t_i} := \{x \in \mathbb{R}^{n+1} : t_i x \in E\}$ converges for $i \to \infty$ in $L^1_{loc}(\mathbb{R}^{n+1})$ to a minimal cone C. Lemma 2.7 ensures that C is nontrivial. Then we get, according to the quoted proposition

$$0 \in \partial^\star C \quad \Longleftrightarrow \quad x_0 = 0 \in \partial^\star E.$$

Assume that C is open. We know [5, Lemma 16.3, p. 184] that C is representable as a subgraph. Thus Theorem 4.5 implies that ∂C is a hyperplane, and therefore we get $\partial C = \partial^\star C$. Using the equivalence from above, we get $x_0 \in \partial^\star E$. Finally, [9, 10] yields that $\partial E \cap \Omega$ is a $C^{1,\lambda}$-manifold. The theorem is proved. □

Theorem 5.2. *Simons' cone*

$$K^{2m} := \left\{ x \in \mathbb{R}^{2m} : \sum_{i=1}^{m} \left(x^i\right)^2 < \sum_{i=m+1}^{2m} \left(x^i\right)^2 \right\}$$

is minimal for $m \geq 4$.

Proof: [5, Theorem 16.4, p. 185] □

Lemma 5.3. *Theorem 5.1 is false in* $\mathbb{R}^9$, *i. e. if we replace the assumption* $E \subset \mathbb{R}^{n+1}$, $n \leq 7$, *by* $E \subset \mathbb{R}^{n+1}$, $n \leq 8$, *we cannot prove* $\partial E \cap \Omega = \partial^\star E \cap \Omega$.

Proof: Let $K := K^8$ be the minimal cone defined in Theorem 5.2. $K \times \mathbb{R} \subset \mathbb{R}^9$ is obviously the subgraph of the function u defined by

$$u(\hat{x}) := \begin{cases} +\infty & : \quad \hat{x} \in K \\ -\infty & : \quad \hat{x} \notin K \end{cases}$$

for all $\hat{x} \in \mathbb{R}^8$. The set $K \times \mathbb{R}$ is minimal [5, Example 16.2, p. 183]. In this example, the measure-theoretical and the reduced boundary differ, we have $\partial(K \times \mathbb{R}) = \partial^\star(K \times \mathbb{R}) \dot{\cup} (\{0\} \times \mathbb{R})$. This equality follows, because the reduced boundary of an almost minimal set $E \subset \mathbb{R}^{n+1}$ consists exactly of those points x of the boundary for which ∂E is a C^1-manifold in a neighborhood of x. □

6. Prescribed Mean Curvature

The following problem is considered in [2]:
In a complete locally conformally flat $(n+1)$-dimensional Riemannian manifold N we look for a closed hypersurface M which is homeomorphic to S^n and has prescribed mean curvature f, $f \in C^{0,1}(N)$, i. e. the equation $H|_M = f(x)$ shall be solved by a hypersurface of class $C^{2,\alpha}$. The hypersurface is looked for in an open connected relatively compact subset Ω of N, which is also regarded - using a diffeomorphism - as a subset of $\mathbb{R}^{n+1}$. The boundary of Ω consists of two components M_1 and M_2, which are given in Euclidean polar coordinates $(x^\alpha)_{0 \leq \alpha \leq n}$, $x^0 \equiv r$, $u_i \in C^{2,\alpha}(S^n, (0, \infty))$, as graphs over S^n: $M_i = \text{graph } u_i|_{S^n} = \{(z, u_i(z)) : z \in S^n\}$. M_1 and M_2 act as barriers, i. e. they satisfy $H|_{M_1} \leq f$ and $H|_{M_2} \geq f$, where the respective unit normal vector (ν^α) is chosen such that the component ν^0 is negative.

In the cited paper it is proven, that under the assumptions stated above, such a hypersurface M exists provided that $n \leq 6$. In this chapter we extend the proof and show that such a hypersurface M exists up to $n = 7$.

Theorem 6.1. *The problem "Find a closed hypersurface* $M \subset \overline{\Omega}$ *of class* $C^{2,\alpha}$ *such that* $H|_M = f$, *which is homeomorphic to* S^n*." has a solution if* $n \leq 7$.

We need some Lemmata:

Lemma 6.2. *The metric product* $S^n \times \mathbb{R}$ *is locally conformally equivalent to* $\mathbb{R}^{n+1} \setminus \{0\}$.

Proof: Let $\tilde{N} = S^n \times \mathbb{R}$ be the metric product of S^n and $\mathbb{R}$. The metric of $\tilde{N}$ is given by

$$d\bar{s}^2_{\tilde{N}} = dt^2 + \sigma_{ij} dx^i dx^j,$$

where $(x^i)_{1\le i\le n}$ are coordinates of S^n and $t \in \mathbb{R}$. Now we identify the manifold $\tilde{N}$ with its image in $\mathbb{R}^{n+1}$ under the diffeomorphism

$$\begin{aligned}\Psi : S^n \times \mathbb{R} \to& \mathbb{R}^{n+1} \setminus \{0\},\\ (x,t) \mapsto& (x, e^t) \equiv (x,r).\end{aligned}$$

Ψ is an isometry, if we equip $\mathbb{R}^{n+1} \setminus \{0\}$ with the metric

$$\frac{1}{r^2} dr^2 + \sigma_{ij} dx^i dx^j.$$

$((x^i)_{1\le i\le n}, r)$ are polar coordinates of $\mathbb{R}^{n+1} \setminus \{0\}$. Now we assume that $(\tilde{N}, d\overline{s}^2_{\tilde{N}}) = (\mathbb{R}^{n+1} \setminus \{0\}, \frac{1}{r^2} dr^2 + \sigma_{ij} dx^i dx^j)$ and deduce

$$\begin{aligned}d\overline{s}^2_{\tilde{N}} =& \frac{1}{r^2} dr^2 + \sigma_{ij} dx^i dx^j\\ =& \frac{1}{r^2}(dr^2 + r^2 \sigma_{ij} dx^i dx^j)\\ =& \frac{1}{r^2} d\overline{s}^2_{\mathbb{R}^{n+1}}.\end{aligned}$$

In the last equality we use the fact that $dr^2 + r^2\sigma_{ij} dx^i dx^j$ is a representation of the standard metric of $\mathbb{R}^{n+1} \setminus \{0\}$ in polar coordinates. This equation yields that $(\tilde{N}, d\overline{s}^2_{\tilde{N}})$ is a locally conformally flat Riemannian manifold. □

Lemma 6.3. *Let $E \subset \mathbb{R}^{n+1}$ be an almost minimal set which is representable as a subgraph over S^n, i. e.*

$$E = \{(\hat{x}, t) \in S^n \times \mathbb{R}^+ \subset \mathbb{R}^{n+1} \setminus \{0\} : t < u(\hat{x})\}$$

is a subgraph in polar coordinates. Assume $u \ge c > 0$. Let $z_0 = (0, z_0^{n+1}) \subset \mathbb{R}^{n+1}$ be an arbitrary point such that $z_0^{n+1} > 0$. If $E_{t_i} = \{y \in \mathbb{R}^{n+1} : z_0 + t_i(y - z_0) \in E\}$ converges for a given sequence $t_i \to 0$, $t_i > 0$, $i \in \mathbb{N} \setminus \{0\}$, in $L^1_{loc}(\mathbb{R}^{n+1})$ to a cone C, then C is representable as a subgraph over $\mathbb{R}^n \equiv \langle z_0 \rangle^\perp$.

Proof: Identify $\mathbb{R}^n$ with $\langle z_0\rangle^\perp$ and introduce orthogonal coordinates. Show that there is no $\hat{x} \in \mathbb{R}^n$ such that $(\hat{x}, t) \in C \cup \partial C$ and $(\hat{x}, \tau) \in \complement C$ with $t > \tau$ or $(\hat{x}, t) \in C$ and $(\hat{x}, \tau) \in \partial C$ with $t > \tau$.

(i) Assume that $z_0^{n+1} = 1$. In order to get an easier representation in coordinates, we translate such that $z_0 = 0$. Then we have

$$E_{t_i} = \{y \in \mathbb{R}^{n+1} : t_i y \in E\}.$$

Now, E is representable as a subgraph over S^n with center $(0, -1)$ and E_{t_i} is representable as a subgraph over S^n with center $(0, -\frac{1}{t_i})$.

(ii) $(\hat{x}, t) \in C$ and $(\hat{x}, \tau) \in \complement C \implies t < \tau$:
Assume that there is a $\hat{x}_0 \in \mathbb{R}^n$, such that $(\hat{x}_0, t_0) \in C \cup \partial C$, $(\hat{x}_0, \tau_0) \in \complement C$ and $t_0 > \tau_0$. Then there exists $\rho > 0$ such that $D_\rho(C, (\hat{x}_0, \tau_0)) = 0$. On the other hand, we have $D_\rho(C, (\hat{x}_0, t_0)) > 2\varepsilon > 0$. Since $E_{t_i} \to C$ in $L^1_{loc}(\mathbb{R}^{n+1})$, we get the inequality $D_\rho(E_{t_i}, (\hat{x}_0, t_0)) > \varepsilon > 0$ for sufficiently large i. Since each E_{t_i} is representable as a subgraph over S^n with center $\left(0, -\frac{1}{t_i}\right)$, we deduce that

$$(0, \infty) \ni s \mapsto D_{s\rho}\left(E_{t_i}, \left(0, -\frac{1}{t_i}\right) + sx\right)$$

is monotone decreasing for any $x \in \mathbb{R}^{n+1}$, because

$$\begin{aligned}
&\chi_{E_{t_i}}\left(\left(0,-\frac{1}{t_i}\right)+s_0y\right)=0,\ s_0>0,\ y\in\mathbb{R}^{n+1}\\
\Longrightarrow &\chi_{E_{t_i}}\left(\left(0,-\frac{1}{t_i}\right)+sy\right)=0 \quad \forall s\ge s_0
\end{aligned}$$

implies

$$s_0^{-n-1}\int\limits_{B_{s_0\rho}\left(\left(0,-\frac{1}{t_i}\right)+s_0x\right)}\chi_{E_{t_i}} \ge s^{-n-1}\int\limits_{B_{s\rho}\left(\left(0,-\frac{1}{t_i}\right)+sx\right)}\chi_{E_{t_i}}\ \forall s\ge s_0.$$

Now, for large i the number $s_i := \frac{\tau_0+\frac{1}{t_i}}{t_0+\frac{1}{t_i}}$ satisfies $0<s_i<1$, and therefore we get for such i

$$\begin{aligned}
0<\varepsilon<&D_\rho(E_{t_i},(\hat{x}_0,t_0))\\
=&D_{1\rho}\left(E_{t_i},\left(0,-\frac{1}{t_i}\right)+1\left(\hat{x}_0,t_0+\frac{1}{t_i}\right)\right)\\
\le&D_{s_i\rho}\left(E_{t_i},\left(0,-\frac{1}{t_i}\right)+s_i\left(\hat{x}_0,t_0+\frac{1}{t_i}\right)\right)\\
=&D_{s_i\rho}(E_{t_i},(s_i\hat{x}_0,\tau_0))
\end{aligned}$$

On the other hand, s_i converges to 1 as i tends to infinity, so we get $D_\rho(C,(\hat{x}_0,\tau_0))\ge \varepsilon>0$; this inequality contradicts $D_\rho(C,(\hat{x}_0,\tau_0))=0$.

(iii) $(\hat{x},t)\in C)$ and $(\hat{x},\tau)\in\partial C\Longrightarrow t<\tau$:
This statement is proved in the same way as (ii).

(iv) So we get for each $\hat{x}\in\mathbb{R}^n$ numbers $t_1,t_2\in[-\infty,+\infty]$ with $t_1\le t_2$ such that

$$\begin{aligned}
\{\hat{x}\}\times(-\infty,t_1)&\subset C,\\
\{\hat{x}\}\times([t_1,t_2]\cap\mathbb{R})&\subset\partial C
\end{aligned}$$

and

$$\{\hat{x}\}\times(t_2,+\infty)\subset \complement C.$$

H^n-almost eyerywhere in $\mathbb{R}^n$ we have $t_1(\hat{x})=t_2(\hat{x})$: Show only that H^n-almost everywhere the inequality $t_2(\hat{x})-t_1(\hat{x})<\frac{1}{k}$ for $k\ge1$ is valid. Define $A_k:=\{\hat{x}\in\mathbb{R}^n: t_2(\hat{x})-t_1(\hat{x})\ge\frac{1}{k}\}$. We get $0=H^{n+1}(\partial C)\ge\frac{1}{k}H^n(A_k)$ and therefore $H^n(A_k)=0$. Define $u(\hat{x}):=t_1(\hat{x})$. We get $C=\operatorname{sub}u$ in $L^1_{loc}(\mathbb{R}^{n+1})$ and u is measurable as C is measurable. □

Lemma 6.4 (C. Gerhardt, according to a Seminar)**.** *Let E be an almost minimal set in $\Omega\Subset S^n\times\mathbb{R}=\tilde{N}=(\tilde{N},\overline{g}_{\alpha\beta})$. Regard $\tilde{N}$ as a subset of $\mathbb{R}^{n+1}$. Then E is also almost minimal in $\Omega\subset\mathbb{R}^{n+1}=(\mathbb{R}^{n+1},\delta_{\alpha\beta})$. The constant λ in the definition of almost minimal remains unchanged for $0<\lambda\le\frac{1}{2}$.*

Proof: **(i)** Almost minimal in a manifold is defined in the same way as in $\mathbb{R}^{n+1}$. Now the balls are geodesic balls and the perimeter is defined using the divergence in the manifold. Since geodesic and Euclidean balls are comparable, i. e. in any relatively compact subset A and for any ρ such that $0<\rho<R(A)$ there exists a constant $c=c(A)>0$ such that $B^{\tilde{N}}_{c\rho}(x)\subset B^{\mathbb{R}^{n+1}}_\rho(x)\subset B^{\tilde{N}}_{\frac{1}{c}\rho}(x)$ for any geodesic

ball in the respective manifold with center $x \in A$. Therefore it suffices to prove the statement for Euclidean balls. From now on we assume that the chosen coordinates are such that the standard metric of $\mathbb{R}^{n+1}$ is represented by the metric tensor $(\delta_{\alpha\beta})_{0\le\alpha,\beta\le n}$.

(ii) Let F be an arbitrary Caccioppoli set in $\tilde{N}$. Let $\varepsilon > 0$, $x_0 \in A$ be arbitrary. Choose $\eta_\varepsilon^\alpha \in C_c^1(B_\rho(x_0))$, $0 \le \alpha \le n$, such that $\overline{g}_{\alpha\beta}\eta_\varepsilon^\alpha\eta_\varepsilon^\beta \le 1$ and

$$\int\limits_{B_\rho(x_0)\subset\tilde{N}} |D\chi_F| \le \varepsilon + \int\limits_{B_\rho(x_0)\subset\tilde{N}} \chi_F D_\alpha \eta_\varepsilon^\alpha.$$

It follows that

$$\begin{aligned}\int\limits_{B_\rho(x_0)\subset\tilde{N}} |D\chi_F| \le& \varepsilon + \int\limits_{B_\rho(x_0)\subset\tilde{N}} \chi_F D_\alpha \eta_\varepsilon^\alpha \\ =&\varepsilon + \int\limits_{B_\rho(x_0)} \chi_F \frac{\partial}{\partial x^\alpha}(\sqrt{g(x)}\eta_\varepsilon^\alpha).\end{aligned}$$

For $x \in B_\rho(x_0)$, we get for sufficiently small $R = R(A)$ and ρ such that $0 < \rho < R$

$$\begin{aligned}\overline{g}_{\alpha\beta}(x_0)\frac{\sqrt{g(x)}}{\sqrt{g(x_0)}}\eta_\varepsilon^\alpha(x)\frac{\sqrt{g(x)}}{\sqrt{g(x_0)}}\eta_\varepsilon^\beta(x) \le& \sup_{y\in A}\sup_{z\in B_\rho(y)}\left(\frac{\sqrt{g(z)}}{\sqrt{g(x_0)}}\right)^2 \cdot \\ &\cdot ((\overline{g}_{\alpha\beta}(x_0) - \overline{g}_{\alpha\beta}(x))\eta^\alpha(x)\eta^\beta(x) \\ &+ \overline{g}_{\alpha\beta}(x)\eta^\alpha(x)\eta^\beta(x))\end{aligned}$$

where the right-hand side can be estimated from above by

$$1 + \rho c(A).$$

These estimates depend especially on the mean value theorem. By $c(A)$ we denote a constant which depends only on A and may change its value from line to line. We have especially $R = R(A) = c(A)$. If we define $\tilde{\eta}_\varepsilon^\alpha = \frac{\sqrt{g(x)}}{\sqrt{g(x_0)}}\eta_\varepsilon^\alpha(x)$, we deduce

$$\begin{aligned}\int\limits_{B_\rho(x_0)\subset\tilde{N}} |D\chi_F| \le& \varepsilon + \int\limits_{B_\rho(x_0)} \chi_F \frac{\partial}{\partial x^\alpha}(\sqrt{g(x_0)}\tilde{\eta}_\varepsilon^\alpha) \\ \le& \varepsilon + (1+\rho c(A)) \cdot \sup\Bigg\{\int\limits_{B_\rho(x_0)} \chi_F \frac{\partial}{\partial x^\alpha}\left(\sqrt{g(x_0)}\eta^\alpha\right) : \\ &\eta^\alpha \in C_c^1(B_\rho(x_0)), 0 \le \alpha \le n, \\ &\overline{g}_{\alpha\beta}(x_0)\eta^\alpha(x)\eta^\beta(x) \le 1 \text{ for } x \in B_\rho(x_0)\Bigg\} \\ \equiv& \varepsilon + (1+\rho c(A)) \cdot P_{(\mathbb{R}^{n+1},\overline{g}(x_0))}(F, B_\rho(x_0)).\end{aligned}$$

As $\varepsilon > 0$ was an arbitrary number, we get

$$\int\limits_{B_\rho(x_0)\subset\tilde{N}} |D\chi_F| \le (1+\rho c(A)) \cdot P_{(\mathbb{R}^{n+1},\overline{g}(x_0))}(F, B_\rho(x_0)).$$

(iv) Let F be again an arbitrary Caccioppoli set in $\tilde N$. Let $0<\rho<R$, $x_0\in A$ and $\eta^\alpha\in C^1_c(B_\rho(x_0))$ such that $\overline{g}_{\alpha\beta}(x_0)\eta^\alpha(x)\eta^\beta(x)\le 1$. For $\tilde\eta^\alpha(x)=\frac{\sqrt{\overline{g}(x_0)}}{\sqrt{\overline{g}(x)}}\eta^\alpha(x)$, we deduce as above

$$\overline{g}_{\alpha\beta}(x)\tilde\eta^\alpha(x)\tilde\eta^\beta(x)\le 1+\rho c(A),$$

and we infer

$$\begin{aligned}\int\limits_{B_\rho(x_0)\subset\tilde N}|D\chi_F|\ge&\frac{1}{1+\rho c(A)}\int\limits_{B_\rho(x_0)}\chi_F\frac{\partial}{\partial x^\alpha}(\sqrt{\overline{g}(x)}\tilde\eta^\alpha(x))\\ \ge&\frac{1}{1+\rho c(A)}\int\limits_{B_\rho(x_0)}\chi_F\frac{\partial}{\partial x^\alpha}(\sqrt{\overline{g}(x_0)}\eta^\alpha(x)).\end{aligned}$$

Now we take the supremum in this inequality over all $\eta^\alpha\in C^1_c(B_\rho(x_0))$ such that $\overline{g}_{\alpha\beta}(x_0)\eta^\alpha(x)\eta^\beta(x)\le 1$. This yields

$$\int\limits_{B_\rho(x_0)\subset\tilde N}|D\chi_F|\ge\frac{1}{1+\rho c(A)}P_{(\mathbb{R}^{n+1},\overline{g}(x_0))}(F,B_\rho(x_0)).$$

(v) The conformal equivalence means $\overline{g}_{\alpha\beta}(x_0)=\vartheta(x_0)\delta_{\alpha\beta}(x_0)$ with $0<\vartheta(x_0)<\infty$. We have

$$P_{(\mathbb{R}^{n+1},\overline{g}(x_0))}(F,B_\rho(x_0))=\varphi(\vartheta(x_0))\int\limits_{B_\rho(x_0)}|D\chi_F|,$$

where φ is a positive and continuous function.

(vi) Now assume again that $E\,\triangle\,F\Subset B_\rho(x_0)$. Using the result above, the inequalities of (iii) and (iv) can be written in the following form:

$$\int\limits_{B_\rho(x_0)\subset\tilde N}|D\chi_F|\le(1+\rho c(A))\varphi(\vartheta(x_0))\int\limits_{B_\rho(x_0)}|D\chi_F|$$

and

$$\int\limits_{B_\rho(x_0)\subset\tilde N}|D\chi_E|\ge\frac{1}{1+\rho c(A)}\varphi(\vartheta(x_0))\int\limits_{B_\rho(x_0)}|D\chi_E|.$$

These inequalities are valid for any $x_0\in A$. From (ii) we deduce the estimate

$$\int\limits_{B_\rho(x_0)\subset\tilde N}|D\chi_E|\le\int\limits_{B_\rho(x_0)\subset\tilde N}|D\chi_F|+K\rho^{n+2\lambda}.$$

Since $\varphi(\vartheta(x_0))$ is a positive and continuous function of x_0, combining the inequalities above yields

$$\int\limits_{B_\rho(x_0)}|D\chi_E|\le(1+\rho\tilde c(A))\int\limits_{B_\rho(x_0)}|D\chi_F|+(1+\rho c(A))c(A)\rho^{n+2\lambda}.\qquad(*)$$

Choose r with $0<r<\rho$ such that

$$\int\limits_{B_\rho(x_0)\setminus B_r(x_0)}|D\chi_E|\le\omega_{n+1}\rho^n.$$

We get the estimate

$$\begin{aligned}\int\limits_{B_\rho(x_0)} |D\chi_{E\cup B_r(x_0)}| \leq & \int\limits_{B_\rho(x_0)\setminus B_r(x_0)} |D\chi_E| + H^n(\partial B_r(x_0)) \\ \leq & \omega_{n+1}\rho^n + (n+1)\omega_{n+1} r^n \\ \leq & (n+2)\omega_{n+1}\rho^n = c(A)\rho^n.\end{aligned}$$

Choosing $F = E \cup B_r(x_0)$, we obtain

$$\int\limits_{B_\rho(x_0)} |D\chi_E| \leq c(A)(\rho^n + \rho^{n+2\lambda}) \leq c(A)\rho^n.$$

We multiply this inequality with $\rho\tilde{c}(A)$, add it to $(*)$ and get

$$(1+\rho\tilde{c}(A)) \int\limits_{B_\rho(x_0)} |D\chi_E| \leq (1+\rho\tilde{c}(A)) \int\limits_{B_\rho(x_0)} |D\chi_F| + c(A)(\rho^{n+2\lambda} + \rho^{n+1}).$$

Now replace λ by $\min\{\lambda, \frac{1}{2}\}$ and conclude

$$\int\limits_{B_\rho(x_0)} |D\chi_E| \leq \int\limits_{B_\rho(x_0)} |D\chi_F| + c(A)\rho^{n+2\lambda}.$$

This inequality states that E is almost minimal in the set $\Omega \subset \mathbb{R}^{n+1}$ equipped with the Euclidean metric. □

Most of the proof of Theorem 6.1 in [2] is independent of the dimension, so the following proof contains mainly the necessary changes to get a proof which is valid up to $n = 7$:

Proof of Theorem 6.1: As in the cited paper we get $u_k \in C^{2,\alpha}(S^n)$, $k \geq 3$, such that

$$H|_{\operatorname{graph} u_k} = f - \gamma e^{-\mu u_k}[u_k - u_{k-1}]$$

and $u_1 \leq u_k \leq u_{k-1}$. These functions converge pointwise, since for a fixed $x \in S^n$, we have a monotone decreasing sequence $u_k(x)$ which is bounded from below. Define u to be the pointwise limit of u_k and further $\varphi_k = \log u_k$ and $\varphi = \log u$. As in [2], we deduce that $E := \operatorname{sub} \varphi = \{(x,t) : t < \varphi(x), x \in S^n\}$ is almost minimal in the metric product $S^n \times \mathbb{R}$. Using Lemmata 6.2 and 6.4 we can regard E an an almost minimal subset in $\mathbb{R}^{n+1}$. Now Lemma 6.3 yields that the blow-up cone C around a point $x \in \partial E$ is representable as a subgraph, thus C is a directed cone, and the regularity theorem implies that C is a half space. Therefore we have $x \in \partial^\star E$ for any $x \in \partial E$. From now on the proof of the theorem in [2] is independent of the dimension and can be used to deduce the final result. □

7. Splitting Theorem

Lemma 7.1. *Let $M \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$ be a measurable set which is directed with respect to $(\hat{x}, t)$. If $\hat{x} \neq 0$, then M is directed with respect to $\hat{x}$.*

Proof: Since $M \times \mathbb{R}$ is directed with respect to $(\hat{x}, t)$, we deduce in view of [11, Theorem 1.3.6]

$$\int\limits_{\hat{B}_\rho(\hat{z})\times[a,b]} \chi_{M\times\mathbb{R}} \geq \int\limits_{\hat{B}_\rho(\hat{z}+\sigma\hat{x})\times[a+\sigma t, b+\sigma t]} \chi_{M\times\mathbb{R}}$$

for $\hat{z} \in \mathbb{R}^n$, $\rho > 0$, $-\infty < a < b < +\infty$, $\sigma > 0$ and $\hat{B}_\rho(\hat{z}) = \{\hat{x} \in \mathbb{R}^n : |\hat{x} - \hat{z}| < \rho\}$. This inequality implies

$$\int\limits_{\hat{B}_\rho(\hat{z})} \chi_M \geq \int\limits_{\hat{B}_\rho(\hat{z}+\sigma\hat{x})} \chi_M,$$

because we have

$$(b-a) \int\limits_{\hat{B}_\rho(\hat{z})} \chi_M = \int\limits_{\hat{B}_\rho(\hat{z})\times[a,b]} \chi_{M\times\mathbb{R}}$$

and an equality of the same kind is valid for the other integrals above. Since $\hat{z} \in \mathbb{R}^n$, $\rho > 0$ and $\sigma > 0$ were chosen arbitrarily, the statement follows. □

The following splitting theorem contains only the case of a singular cone. For regular cones, however, there is no need of such a theorem, because regular cones are half-spaces.

Theorem 7.2 (Splitting Theorem for directed minimal cones)**.**
Let $C \subset \mathbb{R}^{n+1}$ be a singular minimal cone which has k linearly independent directions. Then there is a singular minimal cone $C_0 \subset \mathbb{R}^{n+1-k}$ such that $C = C_0 \times \mathbb{R}^k$ after a suitable rotation and translation of C.

Proof: We may assume that the vertex of the cone and the origin coincide. In view of Lemma 7.1 we have to prove the Theorem only for $k = 1$. Then the statement for $k > 1$ follows by induction. Assume that C is directed with respect to e_{n+1}.

Define $u : \mathbb{R}^n \to [-\infty, +\infty]$ as in [8] to be a measurable function such that $C = \operatorname{sub} u$. Then u is positive homogeneous of degree 1, because C is a cone. According to [5, Theorem 15.5] the set $P := \{\hat{x} \in \mathbb{R}^n : u(\hat{x}) = +\infty\}$ is a minimal set. Since u is positive homogeneous of degree 1 we deduce that P is a minimal cone with its vertex at the origin. Define also $N := \{\hat{x} \in \mathbb{R}^n : u(\hat{x}) = -\infty\}$. N is also a minimal cone with its vertex at the origin.

The case $N = P = \emptyset$ cannot occur: $N = P = \emptyset$ implies $u \in L^\infty_{loc}(\mathbb{R}^n)$ ([5, Proposition 16.7]). According to [4, Theorem 1, p. 317] we deduce $u \in BV(\mathbb{R}^n)$. Finally [3] implies $u \in C^{0,1}(\mathbb{R}^n)$ and $u \in C^\omega(\mathbb{R}^n)$ follows. This would imply that C is regular, a contradiction.

According to Lemma 3.5 we can assume that $P \neq \emptyset$, because in the case $P \neq \emptyset$ we replace C by $-\complement C$ and proceed in the same way as we do now.

We also remark that P is different from a half-space; otherwise we get $P \times \mathbb{R} \subset C$ and in view of [5, Theorem 15.5] $P \times \mathbb{R} = C$, a contradiction to the fact that C is a singular cone. In the same way it can be shown that N is different from a half-space.

$C \neq \mathbb{R}^{n+1}$ implies $P \neq \mathbb{R}^n$. According to the definition of P we deduce that $P \times \mathbb{R} \subset C$. Then [5, Example 16.2] implies that $P \times \mathbb{R}$ is minimal and therefore a singular minimal cone. Now [6, Theorem 2.4], a maximum principle for minimal cones, states, that two minimal cones are equal if they have the same vertex and

one of them contains the other one. We apply this theorem and get $C = P \times \mathbb{R}$. Define $C_0 := P$ and the statement follows. $\square$

Corollary 7.3. *Let $C \subset \mathbb{R}^{n+1}$ be a singular minimal cone with its vertex at the origin. Then we have, after a suitable rotation, $C = C_0 \times \mathbb{R}^k$, where $k \in \mathbb{N}$ is the number of linearly independent directions and C_0 is a singular minimal non-directed cone. In the exeptional case, $k = 0$, we have of course $C = C_0$.*

The splitting theorem contains the Regularity Theorem 4.5.

Corollary 7.4. *Let $C \subset \mathbb{R}^{n+1}$ be a nontrivial minimal cone with its vertex at the origin. Suppose C has k linearly independent directions, $k \in \mathbb{N}$. Then C is a half-space provided that $n + 1 - k \leq 7$ or $H^k(\partial C \setminus \partial^\star C) = 0$.*

Proof: Assume the statement were false and let $C \subset \mathbb{R}^{n+1}$ be a counterexample. Then the splitting theorem yields (after a suitable rotation of C) that $C = C_0 \times \mathbb{R}^k$, where C_0 is a singular minimal cone in $\mathbb{R}^{n+1-k}$.

Since $n + 1 - k \leq 7$ contradicts the non-existence of singular minimal cones up to $\mathbb{R}^7$ ([5]), this case does not occur.

In the second case we deduce that $H^0(\partial C_0 \setminus \partial^\star C_0) \geq 1$, because C_0 is a singular minimal cone. This implies

$$\begin{aligned} H^k(\partial(C_0 \times \mathbb{R}^k) \setminus \partial^\star(C_0 \times \mathbb{R}^k)) =& H^k((\partial C_0 \setminus \partial^\star C_0) \times \mathbb{R}^k) \\ \geq& H^0(\partial C_0 \setminus \partial^\star C_0) \geq 1 \\ >& 0 = H^k(\partial C \setminus \partial^\star C), \end{aligned}$$

contradicting $C = C_0 \times \mathbb{R}^k$. Thus the statement is proved. $\square$

Corollary 7.5. *Let $C \subset \mathbb{R}^{n+1}$ be an open minimal cone with its vertex at the origin. If C is nontrivial and directed with respect to e_{n+1} and there exists a $t > 0$ such that $(0, t) \in \complement C$ or $(0, -t) \in C$, then ∂C is a hyperplane.*

Proof: In view of Lemma 3.5 we can assume that $(0, -t) \in C$. So there exists $\rho > 0$ such that $B_\rho((0, -t)) \subset C$. Thus $u(\hat{x}) > -t$ holds for $|\hat{x}| < \rho$, where u is defined as in Remark 3.9 such that $\operatorname{sub} u = C$. In view of the homogeneity of u, this implies $u(\hat{x}) > -\infty$ for $\hat{x} \in \mathbb{R}^n$. Therefore we deduce that $N = \emptyset$, where the notation from the proof of the splitting theorem has been used and will be used in the rest of this proof. Since $P = \emptyset$ implies the statement as shown in the proof of the splitting theorem, we may assume assume $\emptyset \neq P \neq \mathbb{R}^n$. Following again the proof of the splitting theorem, we deduce that $C = P \times \mathbb{R}$. We choose now $\hat{x} \in \complement P$. Since $\complement P$ is a cone, we have $\tau\hat{x} \in \complement P$ for $\tau > 0$ and $(\tau\hat{x}, -t) \in \complement C$, but $(\tau\hat{x}, -t)$ converges to $(0, -t) \in C$ for $\tau \to 0$. This is a contradiction, because C is an open set. $\square$

Remark 7.6. The assumptions of Corollary 7.5 imply also $\langle \nu, e_{n+1} \rangle < 0$, where ν is the inner unit normal of C.

References

1. E. De Giorgi: *Una estensione del teorema di Bernstein,* Ann. Scuola Norm. Sup. Pisa (3) **19** (1963), 79-85.
2. C. Gerhardt: *Closed Hypersurfaces of prescribed Mean Curvature in locally conformally flat Riemannian Manifolds,* J. Diff. Geom. **48** (1998), 587-613.
3. C. Gerhardt: *On the Regularity of Solutions to Variational Problems in $BV(\Omega)$,* Math. Z. **149** (1976), 281-286.

4. M. Giaquinta, G. Modica, J. Souček: *Cartesian Currents in the Calculus of Variations I,* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Band 37, Springer, Berlin Heidelberg New York, 1998, 736 pages.
5. E. Giusti: *Minimal Surfaces and Functions of Bounded Variation,* Monographs in Math. Vol. 80, Birkhäuser, Boston Basel Stuttgart, 1984, 240 pages.
6. E. Gonzalez, U. Massari, M. Miranda: *On minimal Cones,* Appl. Anal. **65** (1997), 135-143.
7. U. Massari, M. Miranda: *Minimal Surfaces of Codimension One,* Mathmatics Studies Vol. 91, North-Holland, Amsterdam New York Oxford, 1984, 242 pages.
8. M. Miranda: *Superficie minime illimitate,* Ann. Sc. Norm. Sup. Pisa (4) **4** (1977), 313-322.
9. I. Tamanini: *Boundaries of Caccioppoli sets with Hölder-continuous normal vector,* J. Reine Angew. Math. **334** (1982), 27-39.
10. I. Tamanini: *Regularity results for almost minimal oriented hypersurfaces in* $\mathbb{R}^N$*,* Quaderni Dipartimento Math. Univ. Leece **1**, (1984), 92 pages.
11. W. P. Ziemer: *Weakly differentiable Functions,* Graduate Texts in Mathmatics Vol. 120, Springer, Berlin Heidelberg New York, 1989, 308 pages.

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